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Complex Chebyshev Polynomials on Circular Sectors

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All those complex Chebyshev polynomials on circular sectors are given explicitly for which the set of extremal points is a certain set E or a subset of E. The case where z^n is the Chebyshev polynomial on a circular sector is completely characterized. Some graphs representing the absolute value and the argument of certain selected Chebyshev polynomials on circular sectors are presented, demonstrating in particular the heterogeneous behavior of best complex Chebyshev approximations.

1. INTRODUCTION

If B is any nonempty compact set in $\mathbb{C}$ and $n \in \mathbb{N}$, then the polynomial

$$T_n(z) = z^n + a_{n-1}^{(n)} z^{n-1} + a_{n-2}^{(n)} z^{n-2} + \cdots + a_0^{(n)}, \tag{1.1}$$

which is uniquely definied by

$$\| T_n \|_\infty \leqslant \| z^n + b_{n-1} z^{n-1} + b_{n-2} z^{n-2} + \cdots + b_0 \|_\infty \quad \text{for all } b_0, b_1, \ldots, b_{n-1} \in \mathbb{C} \tag{1.2}$$

is called the *Chebyshev polynomial* (for short *T-polynomial*) of degree n with respect to B. The norm $\| \; \|_\infty$ used in (1.2) is the ordinary, uniform norm in $C(B)$, where the functions in $C(B)$ may have complex values.

The definition (1.2) says in other words that $-(a_{n-1}^{(n)} z^{n-1} + a_{n-2}^{(n)} z^{n-2} + \cdots + a_0^{(n)})$ is the best uniform approximation of z^n in the space of all complex polynomials of degree $n-1$ or less.

For any $k \in \mathbb{N}$ we have $\| T_{n+k} \|_\infty \leqslant \| z^k T_n \|_\infty$ therefore

$$B \subset \{z : |z| \leqslant 1\}, \; m \geqslant n \Rightarrow \| T_m \|_\infty \leqslant \| T_n \|_\infty . \tag{1.3}$$

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If we indicate the dependence of T_n on B by $T_n{}^B$ then it is also clear that for any other compact set $C \subset \mathbb{C}$ we have

$$B \subset C \Rightarrow \| T_n{}^B \|_\infty \leqslant \| T_n{}^C \|_\infty . \tag{1.4}$$

If R is a bounded region in $\mathbb{C}$ and B is the closure of R then the fact that $T_n(z)$ is holomorphic implies that the T-polynomials on B and the T-polynomials on the boundary ∂B are identical.

In the literature there are hardly any explicit expressions for complex T-polynomials. One exception is the paper by Faber [1] in which among other things it is shown that all ellipses with foci a, b have the same T-polynomials provided the degree n is fixed. In particular all ellipses with foci ± 1 have T-polynomials which are the T-polynomials on the interval $[-1, 1]$.

In a paper by Opfer [4] an algorithm is presented which is also capable of computing complex approximations. It was applied to the numerical computation of T-polynomials for various domains B (ellipses, square, rectangle) including the case of circular sectors

$$S_\alpha = \{z \in \mathbb{C} : | z | \leqslant 1, | \arg z | \leqslant \alpha\}, \ \alpha \in [0, \pi]. \tag{1.5}$$

The results presented in the sequel were more or less suggested by the computation mentioned.

This paper consists of three parts: Section 2 contains explicit representations of certain T-polynomials on circular sectors; in Section 3 it is shown that z^n is the T-polynomial on S_α if and only if $\alpha \geqslant n\pi/(n+1)$; and in Section 4 some modulus and argument curves of selected T-polynomials are shown demonstrating the irregular pattern of the set of extremal points.

2. T-Polynomials on Circular Sectors

We confine ourselves here to the case $B = S_\alpha$. For brevity we write $T_n{}^\alpha$ instead of $T_n^{S_\alpha}$ as introduced in the last section. Because the S_α are symmetric with respect to the x-axis the $T_n{}^\alpha(z)$ will always have real coefficients $a_0^{(n)}$, $a_1^{(n)}, \ldots, a_{n-1}^{(n)}$.

This and similar statements follow from a symmetry argument that says in effect that the best approximation $p(z)$ of a function $f(z)$ has the same symmetry properties as $f(z)$ (Meinardus [3, Theorem 27]). Here $f(z) = z^n$ has the property $\overline{f(\bar{z})} = f(z)$. The same property for a polynomial $p(z)$ is equivalent to $p(z)$ having only real coefficients.

The case $S_0 = [0, 1]$ therefore is completely real, and the T-polynomials for this case are well known. For theoretical results one should consult Rivlin [5] and for numerical values, Luke [2, p. 462–463].

For $S_\pi = \{z \in \mathbb{C} : |z| \leqslant 1\}$ we have $T_n^\pi = z^n$ for all $n \in \mathbb{N}$. This follows from symmetry arguments alone.

However, as we shall see later there also are angles $\alpha < \pi$ such that $T_n^\alpha(z) = z^n$.

Because of $\| z^n \|_\infty = 1$ the definition (1.2) implies

$$\| T_n^\alpha \|_\infty \leqslant 1 \qquad \text{for all } \alpha \text{ and all } n \in \mathbb{N}. \tag{2.1}$$

For fixed n we conclude from (1.4) and (2.1)

$$\| T_n^\alpha \|_\infty = 1 \Rightarrow \| T_n^\beta \|_\infty = 1 \qquad \text{for all } \beta \geqslant \alpha. \tag{2.2}$$

And from (1.3) and (2.1) we deduce for fixed α

$$\| T_m^\alpha \|_\infty = 1 \Rightarrow \| T_n^\alpha \|_\infty = 1 \qquad \text{for all } n \leqslant m. \tag{2.3}$$

Apparently $\| T_n^\alpha \|_\infty = 1$ happens if and only if $T_n^\alpha(z) = z^n$.

The computations made by Opfer [4] show a rather irregular pattern with respect to the distribution and number of extreme points of $| T_n^\alpha |$, where we call any point $z \in S_\alpha$ an extreme point which satisfies $| T_n^\alpha(z)| = \| T_n^\alpha \|_\infty$. We define

$$E_n^\alpha = \{z \in S_\alpha : | T_n^\alpha(z)| = \| T_n^\alpha \|_\infty\} \tag{2.4}$$

as the set of all extreme points of $| T_n^\alpha(z)|$.

Table I contains the computed number of extreme points for certain cases. In parentheses we list the number of extreme points z with $\operatorname{Im} z \geqslant 0$.

TABLE I

Number of Extreme Points of $T_n^\alpha(z)$

	n			
$\alpha°$	1	2	3	4
0	2(2)	3(3)	4(4)	5(5)
30	3(2)	3(2)	6(4)	6(4)
45	3(2)	4(3)	4(3)	7(4)
60	2(1)	4(3)	4(3)	5(3)
90	$\infty(\infty)$	3(2)	4(2)	6(4)

In the sequel we shall give explicit expressions for those T-polynomials with respect to S_α whose set of extreme points E_n^α is a subset of

$$E = \{0, 1, e^{-i\alpha}, e^{-i\alpha}\}. \tag{2.5}$$

All other cases seem to be very difficult. If $n \geqslant 4$ then E_n^α will contain at least 5 elements. That means that we have to restrict ourselves to the cases $n \leqslant 3$ if we require $E_n^\alpha \subset E$.

In the following we shall use the well-known Kolmogorov criterion (e.g., Meinardus [3, Theorem 18]) which is repeated here for later reference. By P_n we mean the set of all polynomials which have real coefficients and degree n or less.

Kolmogorov criterion: Any polynomial $T_n^\alpha(z)$ of the form (1.1) is the Chebyshev polynomial on S_α if and only if

$$\min_{z \in E_n^\alpha} \operatorname{Re}\{\overline{T_n^\alpha(z)}\, p_{n-1}(z)\} \leqslant 0 \qquad \text{for all } p_{n-1} \in P_{n-1}.$$

THEOREM 1. *$T_1^\alpha(z) = z + a_0^{(1)}$, where*

$$\begin{aligned} -a_0^{(1)} &= 1/(2\cos\alpha) && \text{for } 0 \leqslant \alpha \leqslant 45^\circ, \\ &= \cos\alpha && \text{for } 45^\circ \leqslant \alpha \leqslant 90^\circ, \\ &= 0 && \text{for } \alpha \geqslant 90^\circ. \end{aligned} \tag{2.6}$$

Proof. We show that the Kolmogorov criterion is valid. It reads here

$$\min_{z \in E_1^\alpha} (\operatorname{Re} z + a_0^{(1)})\, b \leqslant 0 \qquad \text{for all } b \in \mathbb{R}, \tag{2.7}$$

where

$$\begin{aligned} E_1^\alpha &= \{0, e^{i\alpha}, e^{-i\alpha}\} && \text{for } 0 \leqslant \alpha \leqslant 45^\circ, \\ &= \{e^{i\alpha}, e^{-i\alpha}\} && \text{for } 45^\circ < \alpha < 90^\circ, \\ &= \{z \in S_\alpha : |z| = 1\} && \text{for } \alpha \geqslant 90^\circ. \end{aligned} \tag{2.8}$$

A simple discussion of all cases the details of which are omitted gives the required result (2.7). ∎

THEOREM 2. *$T_2^\alpha(z) = z^2 + a_1^{(2)}z + a_0^{(2)}$, where*

$$\begin{aligned} -a_1^{(2)} &= \frac{\cos\alpha(1 + 2\sin\alpha)}{1 + \sin\alpha} && \text{for } 15^\circ \leqslant \alpha \leqslant 45^\circ, \\ &= 1 && \text{for } 45^\circ \leqslant \alpha \leqslant 72^\circ, \\ &= 2(1 + \cos\alpha) - [2(1 + \cos\alpha)]^{1/2} && \text{for } 72^\circ \leqslant \alpha \leqslant 120^\circ, \\ &= 0 && \text{for } 120^\circ \leqslant \alpha \leqslant 180^\circ, \end{aligned} \tag{2.9}$$

$$\begin{aligned} a_0^{(2)} &= \frac{\sin\alpha}{1+\sin\alpha} && \text{for } 15° \leqslant \alpha \leqslant 45°, \\ &= \frac{1}{1+2\cos\alpha} && \text{for } 45° \leqslant \alpha \leqslant 72°, \\ &= [2(1+\cos\alpha)]^{1/2} - 1 && \text{for } 72° \leqslant \alpha \leqslant 120°, \\ &= 0 && \text{for } 120° \leqslant \alpha \leqslant 180°. \end{aligned} \tag{2.10}$$

Proof. We observe

$$\begin{aligned} E_2^\alpha &= \{0, e^{\pm i\alpha}\} && \text{for } 15° \leqslant \alpha < 45°, \\ &= \{0, e^{\pm i\alpha}, 1\} && \text{for } 45° \leqslant \alpha \leqslant 72°, \\ &= \{e^{\pm i\alpha}, 1\} && \text{for } 72° < \alpha < 120°, \\ &= \{z \in S_\alpha : |z| = 1\} && \text{for } 120° \leqslant \alpha \leqslant 180° \end{aligned} \tag{2.11}$$

and check the Kolmogorov criterion in every case. The tedious computations are omitted. ▌

Actually the given coefficients for $15° \leqslant \alpha \leqslant 45°$ are even valid for $\alpha_0 \leqslant \alpha \leqslant 45°$, where α_0 is not known but certainly $0° < \alpha_0 < 15°$. Computations show that α_0 is near 12.5°. If $\alpha < \alpha_0$ then E_2^α is no longer a subset of E.

THEOREM 3. *For* $35° \leqslant \alpha \leqslant 60°$ *the T-polynomials* $T_3^\alpha(z)$ *have the following coefficients*:

$$\begin{aligned} -a_0^{(3)} &= \frac{2}{-4\cos^2\alpha + 4\cos\alpha + 7} \\ &\qquad \times [-2\cos^2\alpha + \cos\alpha + 3 - (2\cos\alpha + 2)^{1/2}], \end{aligned} \tag{2.12}$$

$$a_1^{(3)} = \frac{-a_0^{(3)}}{2}(2\cos\alpha + 3) + \cos\alpha, \tag{2.13}$$

$$-a_2^{(1)} = \frac{-a_0^{(3)}}{2}(2\cos\alpha - 1) + \cos\alpha + 1. \tag{2.14}$$

Proof. For the given α we have $E_3^\alpha = E = \{0, 1, e^{i\alpha}, e^{-i\alpha}\}$. Checking the Kolmogorov criterion is not difficult, though lengthy. We omit the details. ▌

Numerical computation indicates that the above given Theorem 3 holds even for a larger intervall $\alpha_0 \leqslant \alpha \leqslant \alpha_1$, with $30° < \alpha_0 \leqslant 35°$, $60° \leqslant \alpha_1 < 65°$.

3. Characterization of the Case $T_n^\alpha = z^n$

As we have already mentioned we know that $T_n^\pi = z^n$ for all $n \in \mathbb{N}$. As we see from the Theorems 1 and 2 we have $T_1^\alpha = z$ for all $\alpha \geqslant \pi/2$ and $T_2^\alpha = z^2$ for all $\alpha \geqslant \frac{2}{3}\pi$. In general we have

THEOREM 4. *$T_n^\alpha(z) = z^n$ if and only if $\alpha \geqslant \alpha_n = n\pi/(n+1)$.*

Proof. For any vector $b = (b_1, b_2, \ldots, b_n) \in \mathbb{R}^n$ and any angle $0 \leqslant \beta \leqslant \pi$ we define

$$c_n(b, \beta) = \sum_{j=1}^{n} b_j \cos(j\beta). \tag{3.1}$$

From the Kolmogorov criterion we know that z^n is the T-polynomial on S_α if and only if

$$\min_{0 \leqslant \beta \leqslant \alpha} c_n(b, \beta) \leqslant 0 \qquad \text{for all } b \in \mathbb{R}^n. \tag{3.2}$$

This in turn is true if and only if $\alpha \geqslant \alpha_n$ as can be seen by the next lemma. ∎

LEMMA 5. *Let $C_n = \{t : t(\beta) = c_n(b, \beta)\}$ be the set of all functions of the form* (3.1) *and $\alpha \geqslant 0$. There exists a trigonometric polynomial $t_n \in C_n$ with the property*

$$t_n(\beta) > 0 \qquad \text{for all } \beta \in [0, \alpha] \tag{3.3}$$

if and only if

$$\alpha < \alpha_n = n\pi/(n+1).$$

Proof. The proof is partitioned into three steps.

Step 1. For any $t \in C_n$ we have

$$t(0) + 2\sum_{k=1}^{n/2} t\left(\frac{2k\pi}{n+1}\right) = 0 \qquad \text{if } n \text{ is even}, \tag{3.4a}$$

or

$$\sum_{k=1}^{(n+1)/2} t\left(\frac{(2k-1)\pi}{n+1}\right) = 0 \qquad \text{if } n \text{ is odd}. \tag{3.4b}$$

In the following we quote these equations just by (3.4) where the appropriate case has to be applied.

For $j \in \{1, 2,..., n\}$ and even n this follows from

$$1 + 2 \sum_{k=1}^{n/2} \cos\left(\frac{2k\pi}{n+1} j\right) = \operatorname{Re}\left\{1 + 2 \sum_{k=1}^{n/2} e^{i[2\pi j/(n+1)]k}\right\}$$

$$= \operatorname{Re}\left\{1 + 2e^{i[2\pi j/(n+1)]} \frac{e^{i[2\pi j/(n+1)](n/2)} - 1}{e^{i[2\pi j/(n+1)]} - 1}\right\}$$

$$= \operatorname{Re} \frac{(-1)^j - \cos[\pi j/(n+1)]}{i \sin[\pi j/(n+1)]} = 0$$

and for odd n this follows from

$$\sum_{k=1}^{(n+1)/2} \cos\left(\frac{(2k-1)\pi}{n+1} j\right) = \operatorname{Re} \sum_{k=1}^{(n+1)/2} e^{i[j\pi/(n+1)](2k-1)}$$

$$= \operatorname{Re}\left\{e^{i[j\pi/(n+1)]} \frac{e^{i[2\pi j/(n+1)][(n+1)/2]} - 1}{e^{i[2\pi j/(n+1)]} - 1}\right\}$$

$$= \operatorname{Re} \frac{(-1)^j - 1}{2i \sin[\pi j/(n+1)]} = 0.$$

Moreover, any trigonometric polynomial $t^{(0)} = \sum_{j=0}^{n} b_j \cos(j\beta)$ with constant term b_0 belongs to the class C_n if it satisfies Eq. (3.4).

To see this we let $t^{(0)} = b_0 + t$ with $t \in C_n$. If $t^{(0)}$ and t both satisfy (3.4) then also the difference $b_0 = t^{(0)} - t$ satisfies (3.4). This implies $b_0 = 0$ and hence $t^{(0)} \in C_n$.

As a consequence we remark that there is no $t \in C_n$ such that

$$t(\beta) > 0 \qquad \text{for all } \beta \in [0, n\pi/(n+1)] \tag{3.5}$$

because that would contradict (3.4). Therefore the existence of a $t_n \in C_n$ with (3.3) implies $\alpha < \alpha_n$.

Step 2. There is a $\tilde{t}_n \in C_n$ with

$$\tilde{t}_n(\beta) \geqslant 0 \qquad \text{for all } \beta \in [0, n\pi/(n+1)], \tag{3.6}$$

namely

$$\begin{aligned} \tilde{t}_n(\beta) &= (1 - \cos\beta)\left(\cos\beta - \cos\frac{n\pi}{n+1}\right) \prod_{k=1}^{(n/2)-1} \left(\cos\beta - \cos\frac{2k\pi}{n+1}\right)^2, \quad n \text{ even}, \\ &= \left(\cos\beta - \cos\frac{n\pi}{n+1}\right) \prod_{k=1}^{(n-1)/2} \left(\cos\beta - \cos\frac{(2k-1)\pi}{n+1}\right)^2, \quad n \text{ odd}. \end{aligned} \tag{3.7}$$

Apparently $\tilde{t}_n(\beta)$ is a polynomial of degree n in $\cos\beta$ which satisfies (3.6) and for which

$$\begin{aligned} \tilde{t}_n\left(\frac{2k\pi}{n+1}\right) &= 0, \qquad k = 0, 1,..., \frac{n}{2} \text{ for } n \text{ even}, \\ \tilde{t}_n\left(\frac{(2k-1)\pi}{n+1}\right) &= 0, \qquad k = 1, 2,..., \frac{n+1}{2} \text{ for } n \text{ odd}. \end{aligned} \tag{3.8}$$

The polynomial $\tilde{t}_n(\beta)$ can be written as a linear combination of 1, $\cos\beta$, $\cos 2\beta$,..., $\cos n\beta$ and because of (3.8) it satisfies (3.4). Thus $\tilde{t}_n(\beta) \in C_n$ as noticed in Step 1.

It is easy to see that $\tilde{t}_n(\beta)$ is defined uniquely by (3.6) up to a positive factor.

Step 3. Let $0 \leqslant \alpha < \alpha_n$. We shall show here the existence of a $t_n \in C_n$ with property (3.3).

For any $\epsilon > 0$ and the polynomials $\tilde{t}_n$ introduced in (3.7) of Step 2 we define

$$t_n(\beta) = \tilde{t}_n(\beta) + \epsilon\tilde{t}_{n-1}(\beta). \tag{3.9}$$

As we see from (3.7) the polynomials $\tilde{t}_n$ and $\tilde{t}_{n-1}$ do not have zeros in common. Thus for any $\epsilon > 0$ (3.6) implies

$$t_n(\beta) > 0 \qquad \text{for } \beta \in [0, \alpha_{n-1}].$$

We assume therefore $\alpha > \alpha_{n-1}$ and investigate $t_n(\beta)$ in $[\alpha_{n-1}, \alpha]$. In this interval $\tilde{t}_n(\beta)$ has a positive minimum; therefore by selecting $\epsilon > 0$ sufficiently small we obtain a $t_n(\beta) > 0$ for all $\beta \in [0, \alpha]$. ▮

Remark. Actually we have proved more than is stated in Theorem 4:

(i) If $T_n^{\alpha}(z) = z^n$ is the nth T-polynomial for S_α then it is the T-polynomial for any compact set C with $S_\alpha \subset C \subset \{z : |z| \leqslant 1\}$, too.

(ii) For $\alpha \in [0, \pi]$ let D_α be the closure of a bounded region in $\mathbb{C}$ with the following properties:

(a) D_α is symmetric with respect to the x-axis,

(b) D_α contains the arc $A_\alpha = \{z \in \mathbb{C} : |z| = 1, |\arg z| \leqslant \alpha\}$,

(c) $D_\alpha - A_\alpha \subset \{z \in \mathbb{C} : |z| < 1\}$.

Then $T(z) = z^n$ is the nth T-polynomial of D_α if and only if $\alpha \geqslant n\pi/(n+1)$. The proof of (i) follows from (1.4), the proof of (ii) follows directly from the proof of Theorem 4 by noticing that only the properties (a), (b), and (c) of S_α were needed to prove that theorem.

4. Modulus and Argument Curves for Certain T-Polynomials on S_α

Because the complex T-polynomials do not behave like their real relatives it seems interesting to exhibit their most prominent features, namely the graphs of the absolute values and arguments of the complex T-polynomials. For this graphic representation we selected $T_n^\alpha(z)$ for $n = 1$ through 4 and $\alpha = 45°$, $90°$ where we plotted $|T_n^\alpha(z)|/\|T_n^\alpha\|_\infty$ rather than $|T_n^\alpha(z)|$. In all cases z varies in the upper half of ∂S_α. The horizontal axis of the given figures represents the arclength of half of ∂S_α starting with $z = 1$, then following the circle up to the vertex at $z = e^{i\alpha}$ and then approaching $z = 0$ on the straight line part (Fig. 1).

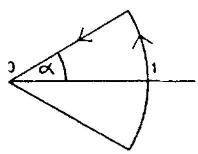

FIG. 1. Circular sector S_α.

The results were obtained by using the algorithm of Opfer [4]. They coincide with the results presented in Section 2.

All curves on Figs. 2–9 (pp. 102–117) were drawn by a BENSON-plotter of the Computation Center of the University of Hamburg.

Acknowledgment

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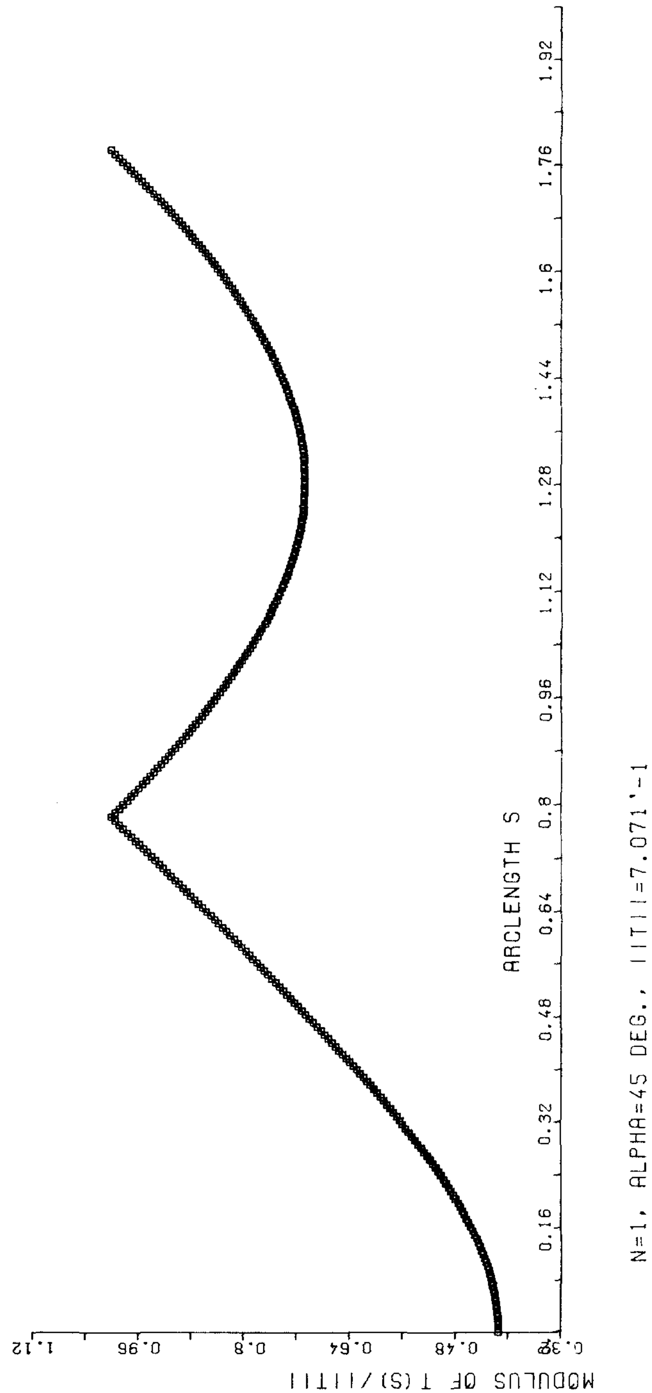
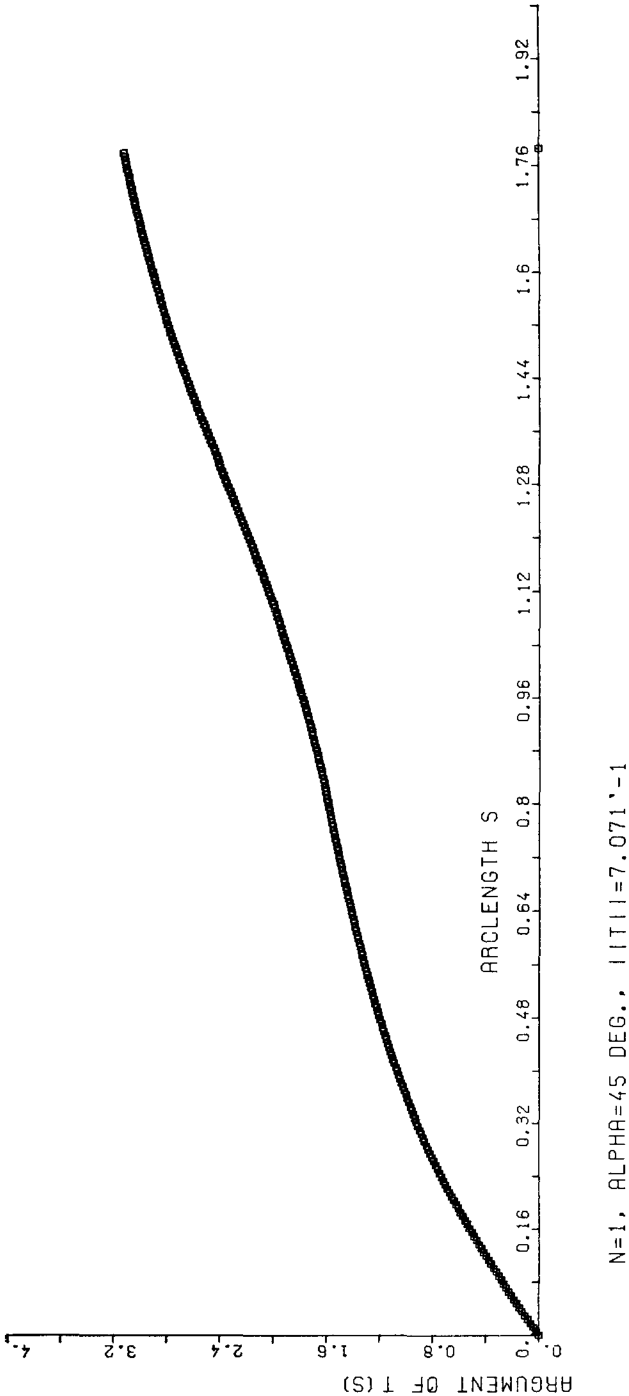
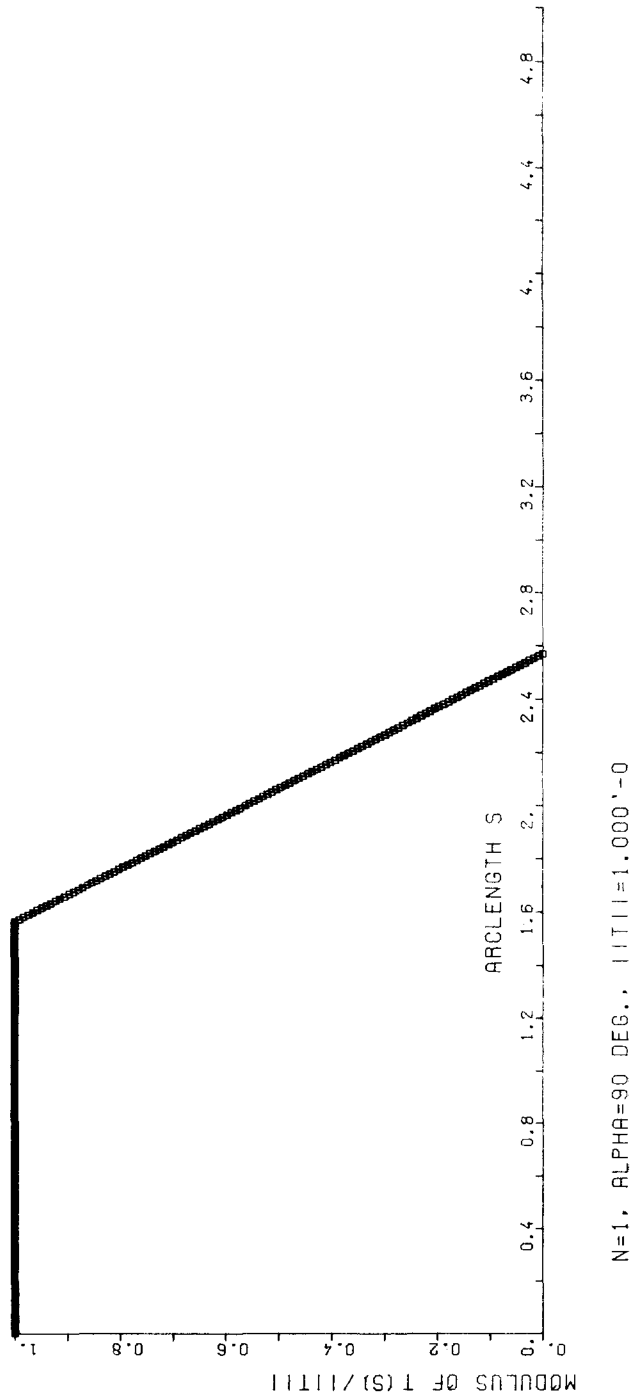

ARCLENGTH S
MODULUS OF T(S)/||T||
N=1, ALPHA=45 DEG., ||T||=7.071'-1
0.16 0.32 0.48 0.64 0.8 0.96 1.12 1.28 1.44 1.6 1.76 1.92
0.32 0.48 0.64 0.8 0.96 1.12



ARCLENGTH S
ARGUMENT OF T (S)
0. 0.16 0.32 0.48 0.64 0.8 0.96 1.12 1.28 1.44 1.6 1.76 1.92
0. 0.8 1.6 2.4 3.2 4.
N=1, ALPHA=45 DEG., ||T||=7.071'-1

FIGURE 2


ARCLENGTH S
0.4
0.8
1.2
1.6
2.
2.4
2.8
3.2
3.6
4.
4.4
4.8
MODULUS OF T(S)/||T||
0.
0.2
0.4
0.6
0.8
1.
N=1, ALPHA=90 DEG., ||T||=1.000'-0

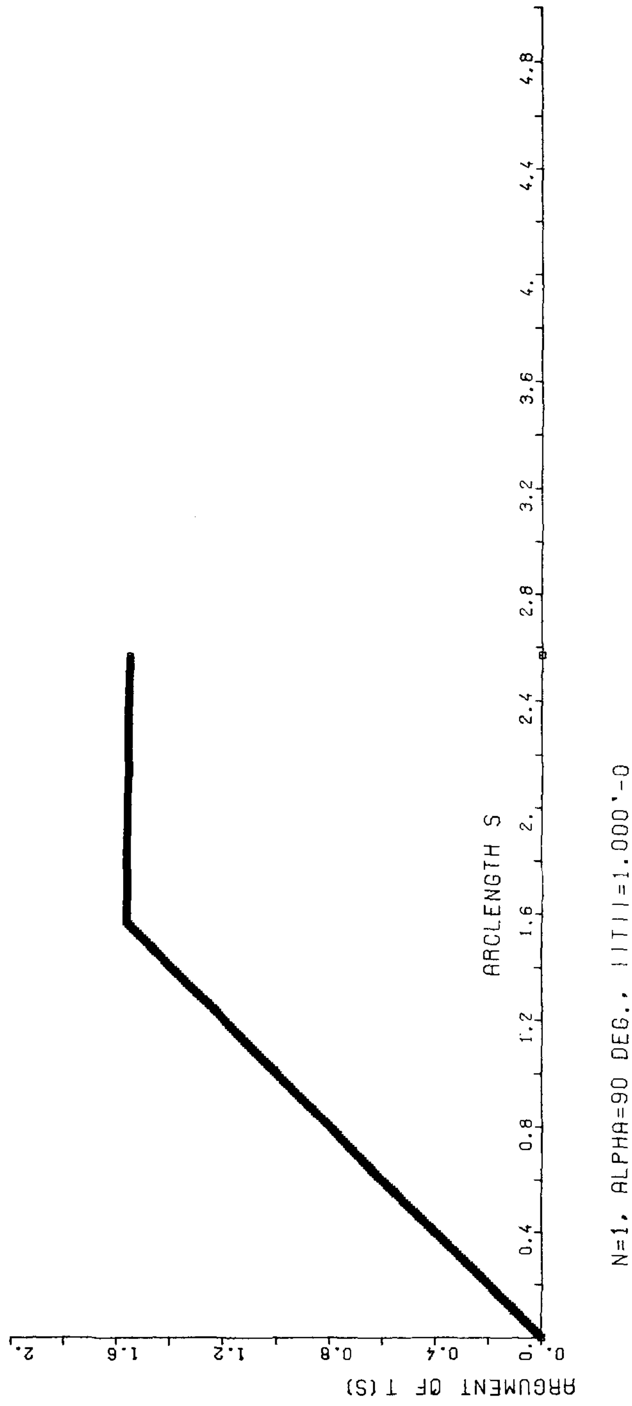
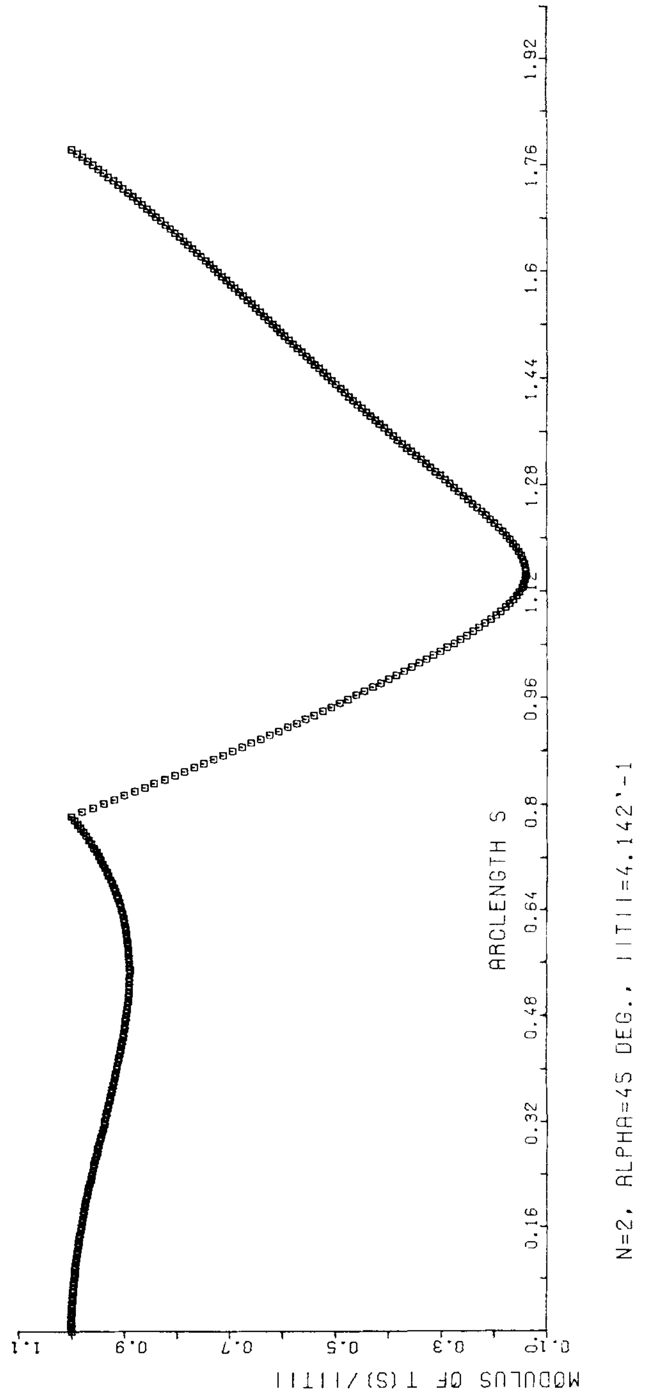

ARCLENGTH S
ARGUMENT OF T(S)
N=1, ALPHA=90 DEG., ||T||=1.000'-0
0. 0.4 0.8 1.2 1.6 2. 2.4 2.8 3.2 3.6 4. 4.4 4.8
0. 0.4 0.8 1.2 1.6 2.

FIGURE 3


MODULUS OF T(S)/||T||
0.1
0.3
0.5
0.7
0.9
1.1
ARCLENGTH S
0.16
0.32
0.48
0.64
0.8
0.96
1.12
1.28
1.44
1.6
1.76
1.92
N=2, ALPHA=45 DEG., ||T||=4.142'-1

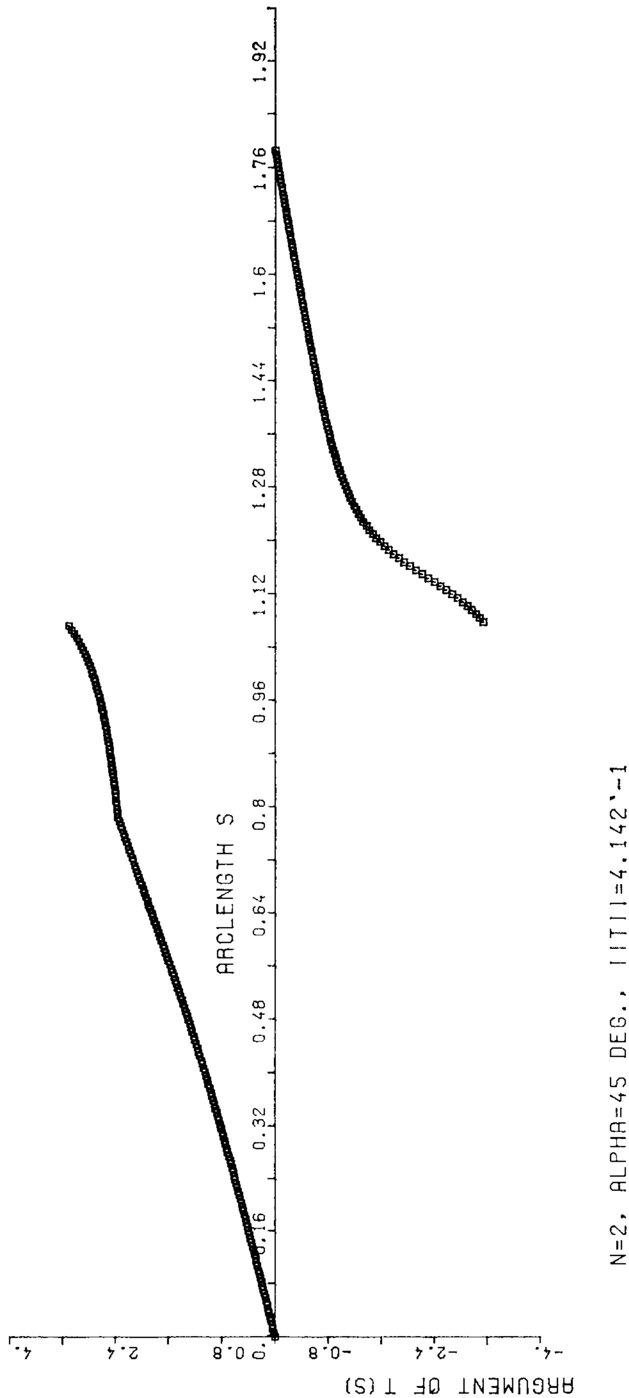
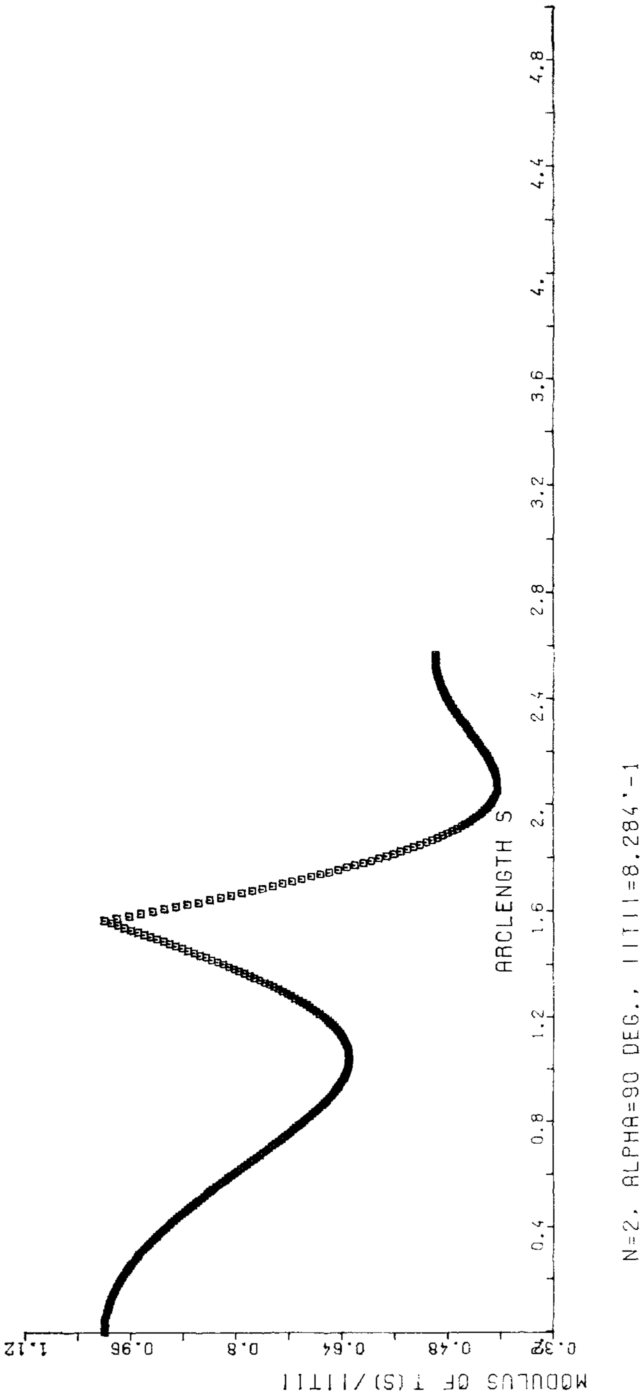

ARCLENGTH S
0.16
0.32
0.48
0.64
0.8
0.96
1.12
1.28
1.44
1.6
1.76
1.92
ARGUMENT OF T (S)
4.
2.4
0.8
0.
-0.8
-2.4
-4.
N=2, ALPHA=45 DEG., ||T||=4.142'-1

FIGURE 4


MODULUS OF T(S)/||T||
ARCLENGTH S
N=2, ALPHA=90 DEG., ||T||=8.284'-1

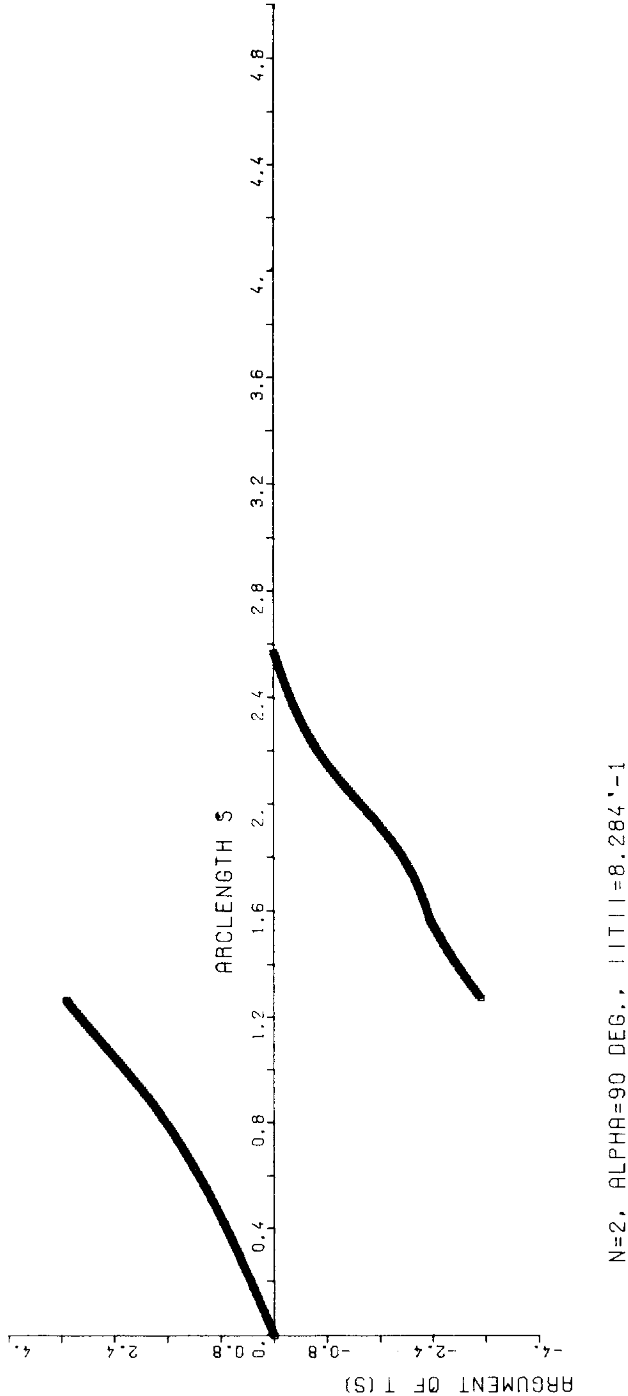
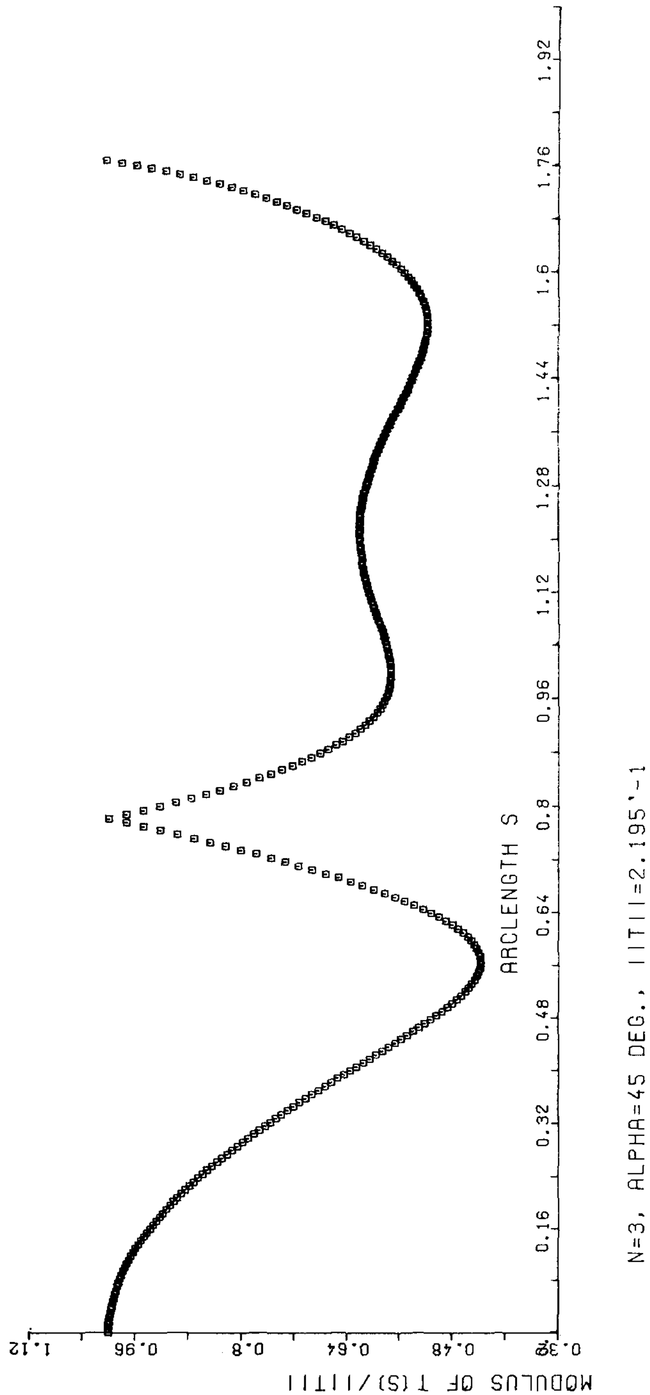

ARCLENGTH S
ARGUMENT OF T (S)
N=2, ALPHA=90 DEG., ||T||=8.284'-1
0.4
0.8
1.2
1.6
2.
2.4
2.8
3.2
3.6
4.
4.4
4.8
4.
2.4
0.8
0.
-0.8
-2.4
-4.

FIGURE 5



ARCLENGTH S
0.16
0.32
0.48
0.64
0.8
0.96
1.12
1.28
1.44
1.6
1.76
1.92
MODULUS OF T(S)/||T||
0.32
0.48
0.64
0.8
0.96
1.12
N=3, ALPHA=45 DEG., ||T||=2.195'-1

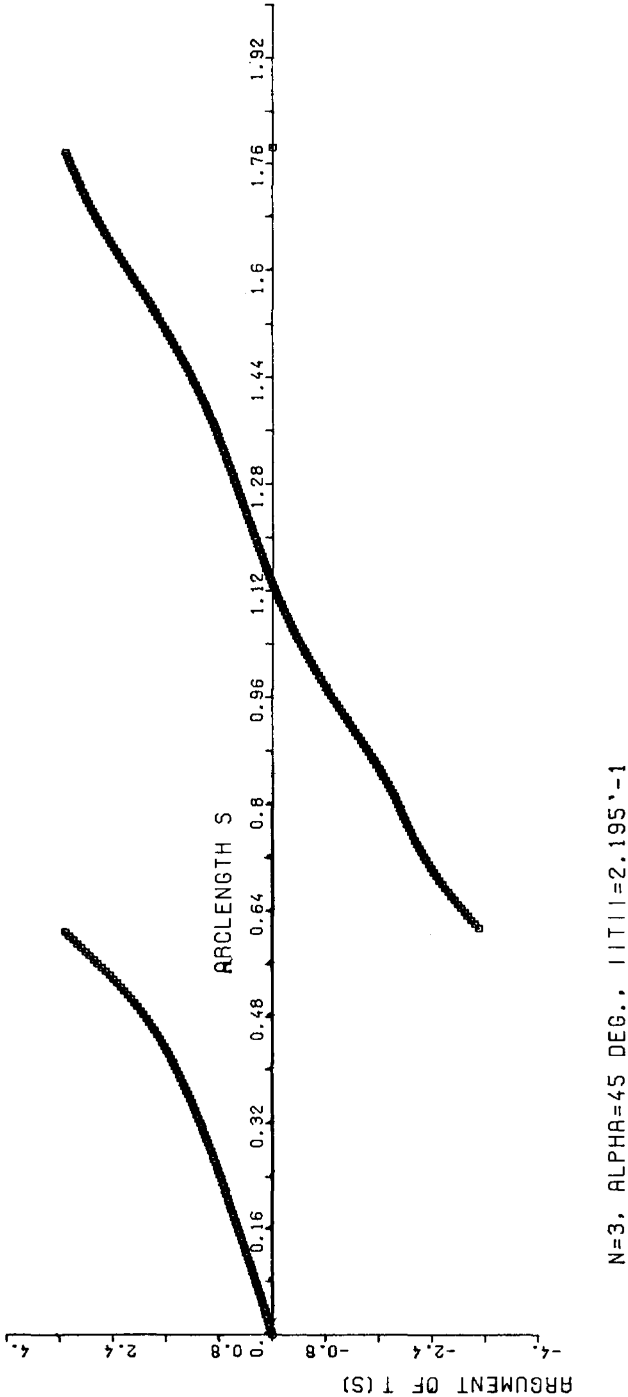
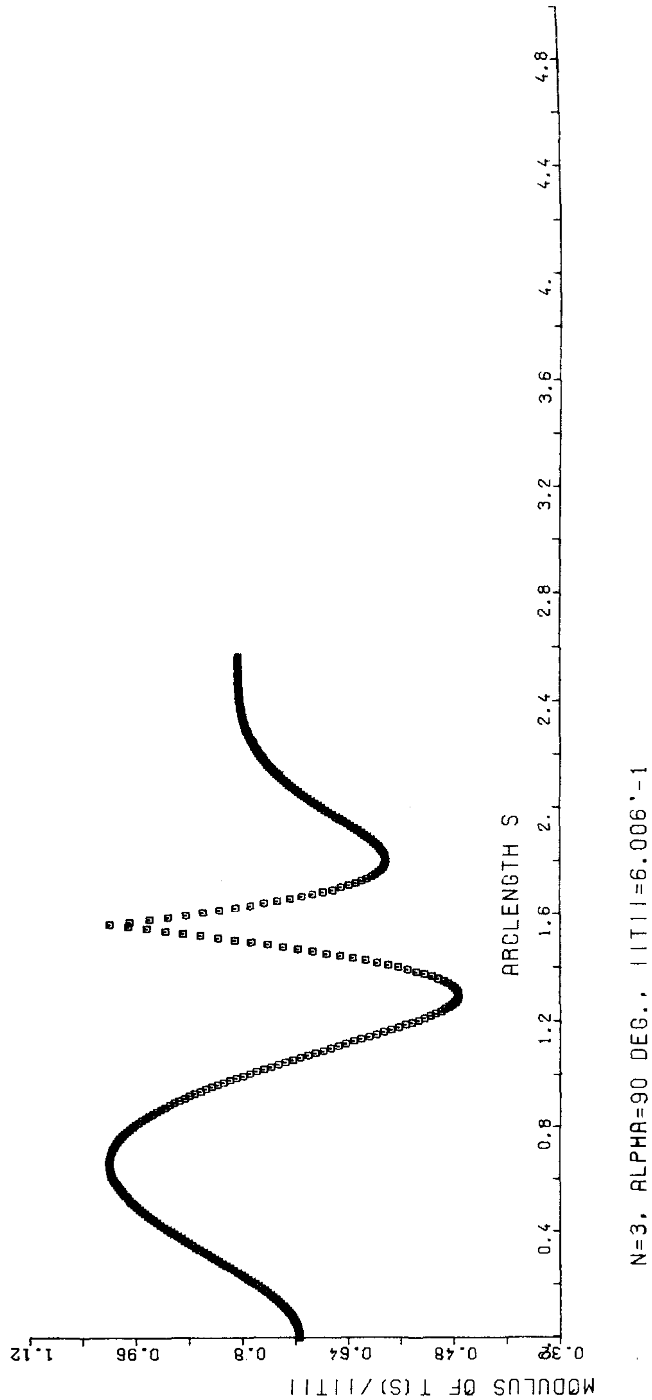
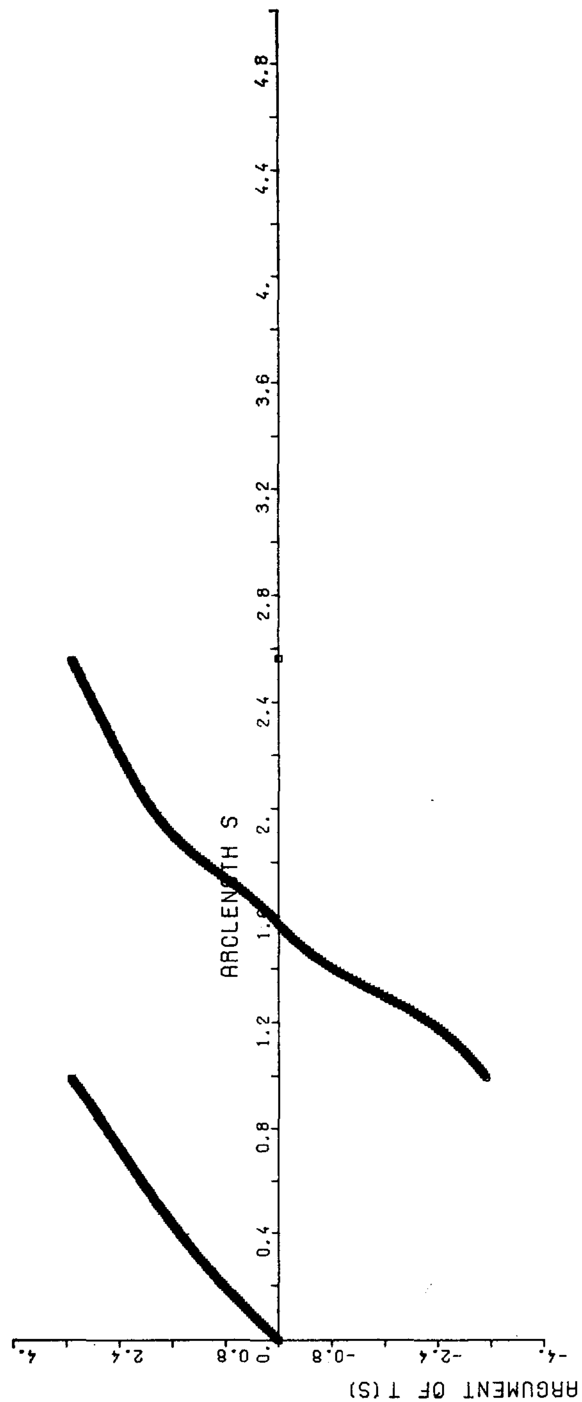


FIGURE 6

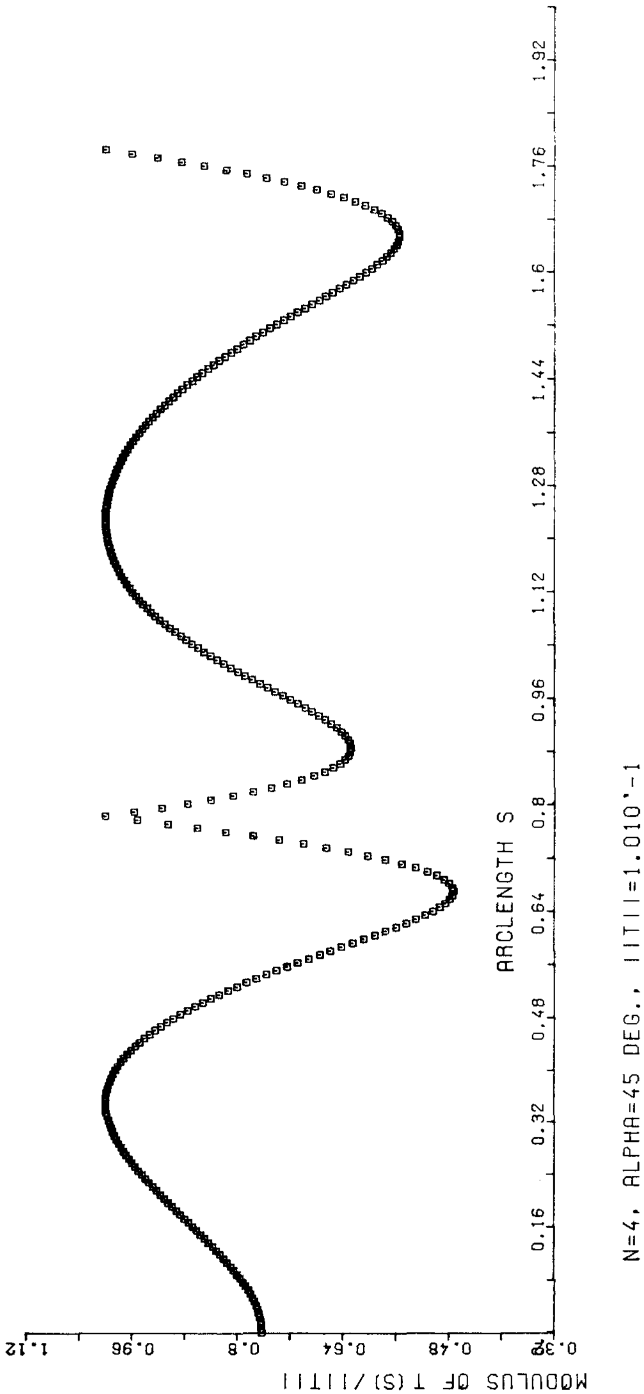


ARCLENGTH S
MODULUS OF T(S)/||T||
N=3, ALPHA=90 DEG., ||T||=6.006'-1
0.4
0.8
1.2
1.6
2.
2.4
2.8
3.2
3.6
4.
4.4
4.8
0.32
0.48
0.64
0.8
0.96
1.12



N=3, ALPHA=90 DEG., ||T||=6.006'-1

FIGURE 7


ARCLENGTH S
0.16
0.32
0.48
0.64
0.8
0.96
1.12
1.28
1.44
1.6
1.76
1.92
MODULUS OF T(S)/||T||
0.32
0.48
0.64
0.8
0.96
1.12
N=4, ALPHA=45 DEG., ||T||=1.010'-1

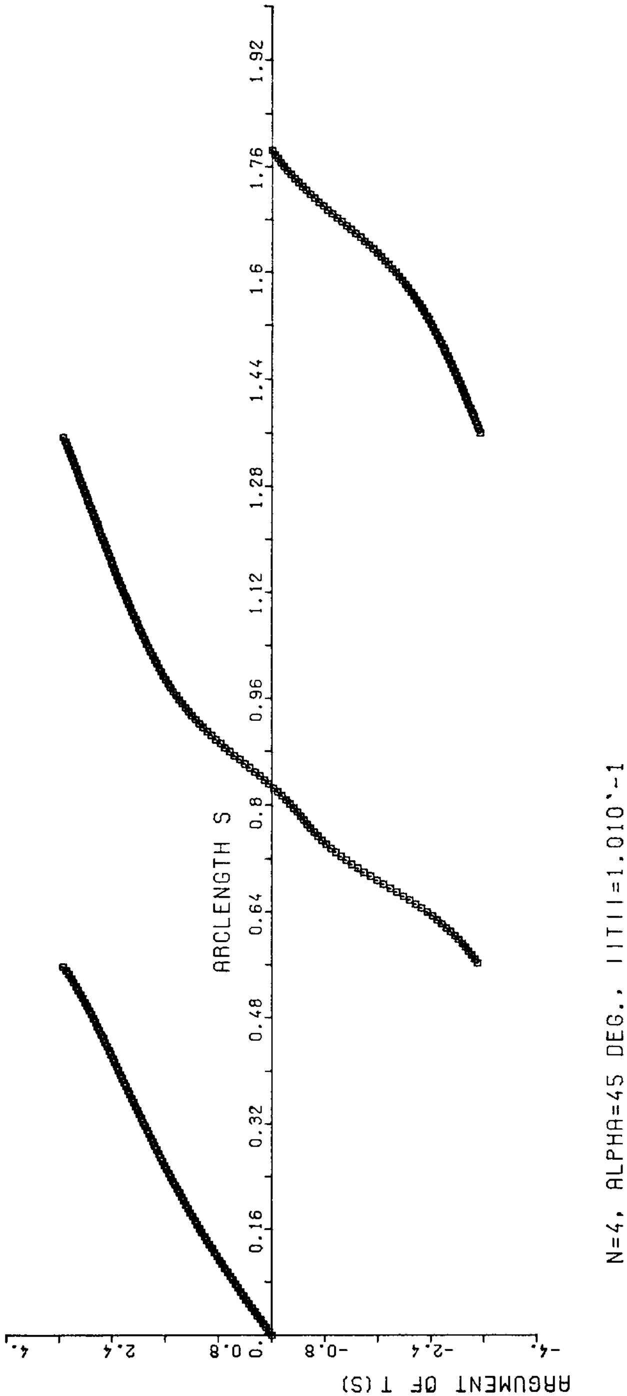
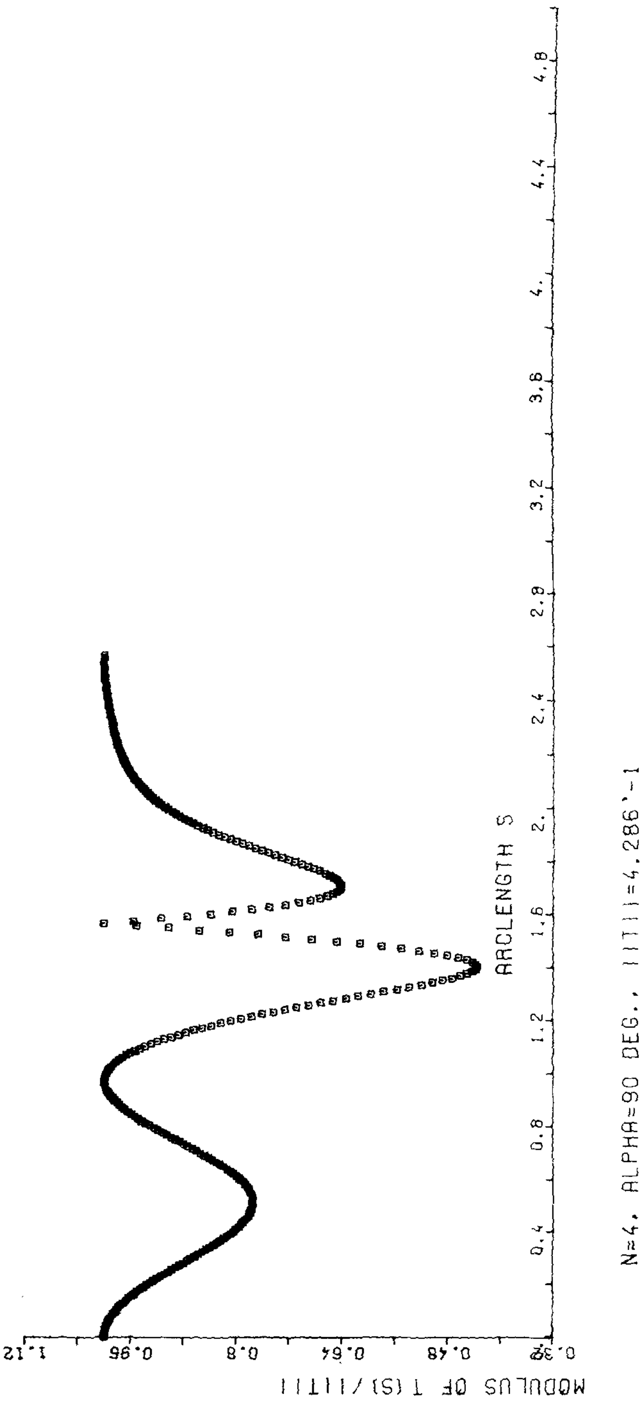
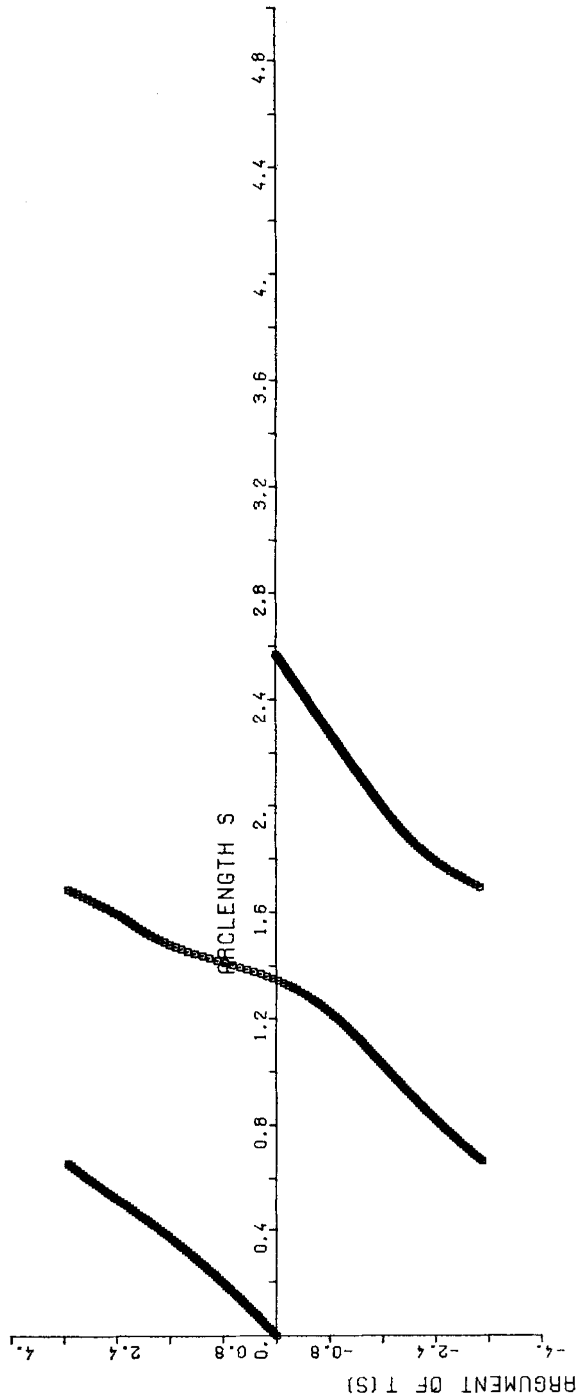

ARCLENGTH S
0.16
0.32
0.48
0.64
0.8
0.96
1.12
1.28
1.44
1.6
1.76
1.92
ARGUMENT OF T (S)
4.
2.4
0.8
0
-0.8
-2.4
-4.
N=4, ALPHA=45 DEG., ||T||=1.010'-1

FIGURE 8


MODULUS OF T(S)/||T||
1.12
0.96
0.8
0.64
0.48
0.32
0.4
0.8
1.2
1.6
2.
2.4
2.8
3.2
3.6
4.
4.4
4.8
ARCLENGTH S
N=4, ALPHA=90 DEG., ||T||=4.286'-1



N=4, ALPHA=90 DEG., ||T|| = 4.286'-1

FIGURE 9

REFERENCES

1. G. FABER, Über Tschebyscheffsche Polynome, *J. Reine Angew. Math.* **150** (1920), 79–106.
2. Y. L. LUKE, "Mathematical Functions and their Approximations," Academic Press, New York/San Francisco/London, 1975.
3. G. MEINARDUS, "Approximation of Functions: Theory and Numerical Methods," Springer-Verlag, Berlin/Heidelberg/New York, 1967.
4. G. OPFER, An algorithm for the construction of best approximations based on Kolmogorov's criterion, *J. Approximation Theory* **23** (1978), 299–377.
5. T. J. RIVLIN, "The Chebyshev Polynomials," Wiley, New York/London/Sydney/Toronto, 1974.